

Optimal dynamics for quantum-state and entanglement transfer through homogeneous quantum wires

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(Dated: June 8, 2010)

It is shown that effective quantum-state and entanglement transfer can be obtained by inducing a coherent dynamics in quantum wires with homogeneous intrawire interactions. This goal is accomplished by tuning the coupling between the wire endpoints and the two qubits there attached, to an optimal value. A general procedure to determine such value is devised, and scaling laws between the optimal coupling and the length of the wire are found. The procedure is implemented in the case of a wire consisting of a spin- $\frac{1}{2}$ XY chain: results for the time dependence of the quantities which characterize quantum-state and entanglement transfer are found of extremely good quality and almost independent of the wire length. The present approach does not require *ad hoc* engineering of the intrawire interactions nor a specific initial pulse shaping, and can be applied to a vast class of quantum channels.

One of the most commonly requested conditions in quantum communication and computation protocols is that two distant parties, typically Alice and Bob, share a couple of entangled qubits. When the physical objects encoding the qubits can travel, as in the case of optical photons, the above goal can be accomplished by creating the entangled couple in a limited region of space and then letting the qubits fly where necessary. On the other hand, when qubits are realized via intrinsically localized physical objects, as in the case of $S = \frac{1}{2}$ spins or atomic systems, a different strategy must be adopted (see for instance Ref. 1 and references therein). One such strategy is the following: first, two neighboring qubits (A and A') are prepared in an entangled state, by means of a short-range interaction; then, the mixed state of one of the two qubits (say A) is transferred to a third distant qubit via a quantum channel. If state-transfer is perfect the procedure results in a pair of distant entangled qubits A' and B, as requested.

Aim of this paper is to set a general framework where such strategy can be successful. In particular, (i) we define a procedure for controlling such dynamics, and hence the quality of the transmission process, by specific operational settings; (ii) we show that the quality of the quantum-state and entanglement transfer is not substantially affected by the length of the wire; (iii) we apply the procedure to the spin- $\frac{1}{2}$ XY chain and show that high-quality quantum-state and entanglement transfer are obtained.

Let us first recall that for the strategy depicted above to make sense, one has to equip oneself with a quantum channel capable of transferring mixed states. How to obtain such a channel is the problem to which many authors have proposed different solutions [1–7], some based on the idea of engineering the channel itself, by the specific design of its internal interactions, others on that of intervening on the initialization process, by preparing

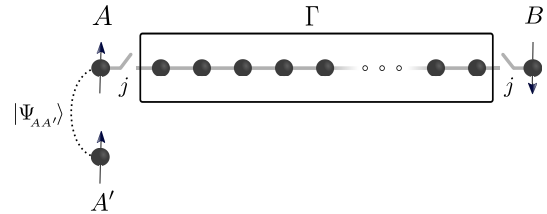


FIG. 1: The endpoints of a quantum wire Γ are coupled to the qubits A and B, via a switchable interaction j ; A can be entangled with an external qubit A'.

the wire in a configuration found to serve the purpose. In both cases, a severe external action on the physical system is required.

Here a different point of view is adopted: the purpose is that of devising conditions for an optimal dynamics to occur, where by ‘optimal dynamics’ we mean a time evolution of the state of the channel such that the mixed state of A is transferred with high fidelity through the channel to a distant party B. The main guideline is the fact that excitations characterized by a linear dispersion play a crucial role in determining effective transfer of quantum states. Since this idea has been put forward [3, 8, 9], it had to face the unfortunate evidence that a quantum channel with a linear dispersion relation over the whole Brillouin zone is not only difficult to design [4], but perhaps just a chimera to realize, so far. On the other hand, most physical systems are characterized by excitations whose dispersion relation has zones of linearity, typically near inflection points: if we were able to induce a dynamical evolution of the channel essentially ruled by excitations belonging to such zones, a coherent propagation should result, and an effective transfer of quantum states consequently obtained. Let us hence focus on how to induce such dynamical evolution.

We consider (see Fig. 1) two separated parties, Alice and Bob, each owning a qubit, A and B, respectively. Each qubit may be subjected to a local interaction, $h_Q \mathcal{H}_A$ and $h_Q \mathcal{H}_B$, respectively, with h_Q possibly tunable. The quantum channel is realized by an extended physical system Γ , hereafter called *quantum wire*, which is here assumed to be made of N interacting particles on a discrete lattice. The internal dynamics of the wire is ruled by the Hamiltonian \mathcal{H}_Γ . The endpoints of the wire can be put in contact with the qubit A (B) via a switchable interaction $j\mathcal{H}_{A\Gamma}$ ($j\mathcal{H}_{B\Gamma}$). The overall system has mirror symmetry and the Hamiltonian which rules its dynamics is $\mathcal{H} = \mathcal{H}_\Gamma + j(\mathcal{H}_{A\Gamma} + \mathcal{H}_{B\Gamma}) + h_Q(\mathcal{H}_A + \mathcal{H}_B)$. Before the process starts A, B and Γ do not interact with each other; the wire is in its ground state $|\Omega_\Gamma\rangle$, while Alice and Bob prepare their qubits in the initial states $\rho_A(0)$ and $\rho_B(0)$, respectively. At time $t=0$ the interaction between each qubit and the wire is switched on.

We assume that: (i) \mathcal{H} can be written (either exactly or via a reasonable approximation) as a quadratic form of $N+2$ local, either fermionic or bosonic, operators $\{\eta_i, \eta_i^\dagger\}$, where the index $i=0, \dots, N+1$ labels the sites of the lattice; (ii) the bulk of the wire is translation symmetric. Condition (i) guarantees the existence of a unitary transformation $\{\eta_i, \eta_i^\dagger\} \rightarrow \{\eta_k, \eta_k^\dagger\}$, which diagonalizes the total Hamiltonian, $\mathcal{H} = \sum_k \omega_k \eta_k^\dagger \eta_k$. Condition (ii) entails that the above transformation is close to a Fourier transform when i corresponds to sites in the bulk of the wire, so that k can be approximately considered as a quasi-momentum. For the sake of clarity, let us consider the qubits A and B as initially prepared in the pure states $|\alpha\rangle$ and $|\beta\rangle$, respectively; the overall system at time $t=0$ is then described by $|\Psi_0\rangle = |\alpha\rangle|\Omega_\Gamma\rangle|\beta\rangle$, which generally is a non-equilibrium state. Its dynamics is ruled by quasi-particle excitations with density in k -space $n(k) = \langle \Psi_0 | \eta_k^\dagger \eta_k | \Psi_0 \rangle$, which, in their turn, evolve according to \mathcal{H} . If the dispersion relation were linear, $\omega_k \simeq \omega_0 + vk$, the bulk dynamics would consist in a coherent wavepacket traveling from A to B (and vice versa) at velocity v along the wire. Under our further assumption of mirror symmetry, such wavepacket would perfectly rebuild the initial state of A on the qubit B, after a time $t \simeq N/v$.

However, dispersion relations of real interacting systems are typically non linear, and display a much more complex dependence on the wavevector k (see for instance Fig. 2). On the other hand, if we cannot deal with a most unusual linear dispersion, yet we can act on the distribution $n(k)$ by varying the coupling between the qubits and the wire, or other parameters entering the Hamiltonian of the qubits, or their initial states $|\alpha\rangle$ and $|\beta\rangle$.

Suppose then that we can adjust the distribution $n(k)$ to make it peaked around an inflection point k_0 of the dispersion relation, thus setting the most relevant excitations for the dynamics, as those whose energies can be written as

$$\omega_k \simeq \omega_{k_0} + v(k-k_0) + \frac{2a}{3}(k-k_0)^3, \quad (1)$$

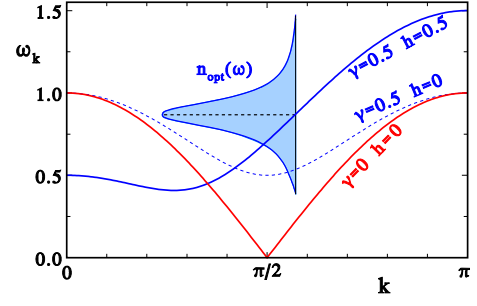


FIG. 2: Dispersion relation of the spin- $\frac{1}{2}$ XY chain, for different interaction parameters. The bell-shaped curve edging the shaded area is the optimal density of excitation vs. ω , as obtained for $\gamma = \gamma_{0,N} = 0.5$, and $h = 0.5$.

where the main dispersive term is retained. Further suppose that we can set the width of the distribution $n(k)$: be $1/(4\sigma^2)$ its variance. It can be shown [10] that, under the above conditions, the dynamics of the wire is still described by a traveling spacial wavepacket, centered at $x(t) = (v + a\sigma^{-2})t$, with a variance that increases from the initial value $\sigma^2(0) = \sigma^2$ as $\sigma^2(t) = \sigma^2 + a^2 t^2 / (2\sigma^4)$.

We hence state that the optimal dynamics is obtained by choosing σ^2 so as to minimize $\sigma(t)$ at the arrival time $t \simeq N/v$, leading to

$$\sigma_{\text{opt}} = \sqrt[3]{|a|t} \simeq (|a|N/v)^{1/3}. \quad (2)$$

As stated above, the form of the dispersion relation is essentially designed by the wire Hamiltonian, the position of the peak of $n(k)$ can be set by acting on h_Q , while σ is expected to depend on the coupling between the qubits and the wire, whose energy scale is j .

Based on the above reasoning, we define the following procedure for inducing an optimal dynamics: Fix the Hamiltonian parameters of the wire so as to grant the existence of a wide enough interval in k with almost linear dispersion [11], as described by Eq.(1); tune h_Q so as to peak $n(k)$ at an inflection point of ω_k ; then set σ to its optimal value Eq. (2) by varying j . The value of the coupling corresponding to σ_{opt} will be hereafter referred to as the *optimal coupling*, j_{opt} . It is worth mentioning that the optimization of σ can also be obtained by creating a finite-size excitation in space as the starting state [3, 5, 12, 13]; however, this strategy results in a much less versatile procedure. The weak dependence on N embodied in Eq. (2) is expected to give small losses for transmissions over long wires. In fact, the actual time t_N needed for the packet to travel along the wire, implicitly defined by $x(t_N) = N$, is found to depend on N according to $v t_N \simeq N - \frac{\text{sgn } a}{2} \sigma_{\text{opt}}$. It is of absolute relevance that $\sigma(t_N) \simeq \sqrt{3/2} \sigma_{\text{opt}}$, i.e., the final optimal packet is just about 22% wider than at start, irrespectively of the wire length.

Let us now describe a specific implementation of the procedure described above. We consider a system defined on a one-dimensional discrete lattice of $N+2$ sites,

labeled by the index $i = 0, 1, \dots, N+1$. The qubits A and B sit at sites 0 and $N+1$, respectively. The wire is physically realized by a chain of N interacting $S = \frac{1}{2}$ spins, taking up sites from 1 to N . Neighboring spins interact via a Heisenberg Hamiltonian of the XY type, and are possibly subjected to an external magnetic field. The exchange interaction and the magnetic field are assumed homogeneous along the wire. The total Hamiltonian is

$$\begin{aligned} \mathcal{H} = & - \sum_{i=1}^{N-1} (1+\gamma) S_i^x S_{i+1}^x + (1-\gamma) S_i^y S_{i+1}^y - h \sum_{i=1}^N S_i^z \\ & - j \sum_{i=0, N} \left[(1+\gamma_i) S_i^x S_{i+1}^x + (1-\gamma_i) S_i^y S_{i+1}^y \right] \\ & - h_Q (S_0^z + S_{N+1}^z), \end{aligned} \quad (3)$$

where the exchange energy for the intrawire interaction has been taken as the reference energy scale and hence set to unity. Mirror symmetry implies $\gamma_N \equiv \gamma_0$. An isotropic exchange interaction in Eq. 3, $\gamma = \gamma_0 = 0$, defines the so called XX model. A Jordan-Wigner transformation, casts the Hamiltonian (3) into a quadratic form of $N+2$ interacting fermionic operators, $\{\eta_i, \eta_i^\dagger\}$, each defined on a site of the lattice. A further Bogolubov transformation diagonalizes the Hamiltonian, $\mathcal{H} = \sum_k \omega_k \eta_k^\dagger \eta_k + E_0$, where E_0 is the ground state energy, whereas η_k^\dagger (η_k) are fermionic operators which create (annihilate) excitations of energy ω_k [14].

The optimization procedure described above begins with the analysis of the dispersion relation in the infinite chain limit, $\omega_k = [(h - \cos k)^2 + \gamma^2 \sin^2 k]^{1/2}$, displayed in Fig. 2. One can easily spot the existence of more or less wide regions of linearity in the neighborhood of the inflection point(s) k_0 , where Eq. (1) holds. By tuning $h_Q \simeq \omega_{k_0}$, the peak of $n(k)$ is made to sit at k_0 . The optimal coupling j_{opt} is then numerically determined so as to fulfill Eq. (2). In fact, the dispersion relation strongly depends on the parameters of the wire Hamiltonian, γ and h : in particular, the region of linear dispersion sensibly shrinks as the anisotropy γ increases, which might make the wire to be useless; however, the linear region can be extended again by increasing the field h : therefore, one can act on the latter parameter so as to fulfil the conditions for optimal dynamics. For example, in the extreme case of the Ising chain ($\gamma = 1$) for $h = 0$ the dispersion relation becomes flat and does not allow for propagation; however, a wide linear region can yet be obtained by applying a finite h on Γ .

Let us now give a flash upon the resulting dynamics of the overall system. To this aim, we set $\gamma = \gamma_0 = 0$ (XX model) and $h = 0$, as this choice makes more analytical expressions available, which in turn allows for a more detailed analysis. The inflection point $k_0 = \pi/2$ corresponds to $\omega_{k_0} = 0$ and we hence set $h_Q = 0$.

In Fig. 3 we show the time evolution of the magnetization parallel to the quantization axis all along the wire, $\langle S_i^z(t) \rangle, i = 0, 1, \dots, N, N+1$, for $N = 50$ and initial states of the qubits $|\alpha\rangle = |\beta\rangle = |\uparrow\rangle$. The upper panel

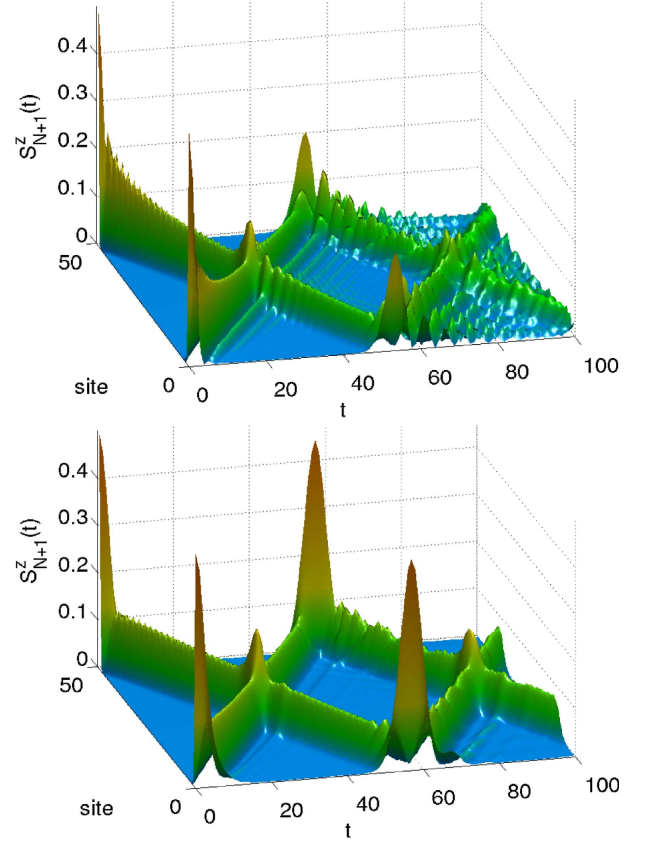


FIG. 3: Time evolution of the on-site z -magnetization for $N = 50$ and an initial state $|\uparrow\rangle|\Omega_T\rangle|\uparrow\rangle$; the qubit-wire coupling is $j = 1$ (top) and $j = j_{\text{opt}} = 0.58$ (bottom).

corresponds to a generic value of the coupling, $j = 1$. The lower panel is for $j = j_{\text{opt}} = 0.58$: this value is determined exactly by exploiting analytical expressions holding for the XX model [15]. The difference from the previous panel is striking: indeed, the dynamics of the wire is essentially ruled by the nondispersive propagation of wavepackets, and results in $\langle S_B^z(t) \rangle$ being, at the arrival time $t \simeq N$, an almost perfect reproduction of the initial magnetization $\langle S_A^z(0) \rangle$ of the qubit A. These results do also confirm that our prescription for the above coherent propagation to occur, i.e., for the determination of j_{opt} , is correct.

But does this peculiar dynamical evolution also affect the quality of quantum-state transfer? To answer this question, we now deal with the time evolution of quantities which are used to monitor such quality. In particular, we analyze the quantum-state transfer process in terms of the fidelity between an initial state $|\alpha\rangle$ of A and the evolved state $\rho_B(t)$ of B, $\mathcal{F}_{AB}(t) = \langle \alpha | \rho_B(t) | \alpha \rangle$, where initially $\rho_B(0) = |\uparrow\rangle\langle\uparrow|$. As for the entanglement, we refer to the time evolution of the concurrence between A' and B, $\mathcal{C}_{BA'}(t) \equiv \mathcal{C}(\rho_{BA'}(t))$ [16]. In the case of the XX model we can also evaluate the *minimum* fidelity over all possible $|\alpha\rangle$, hereafter indicated by $\mathcal{F}_{AB}^{\text{min}}(t)$, which allows us to put forward further conclusions about

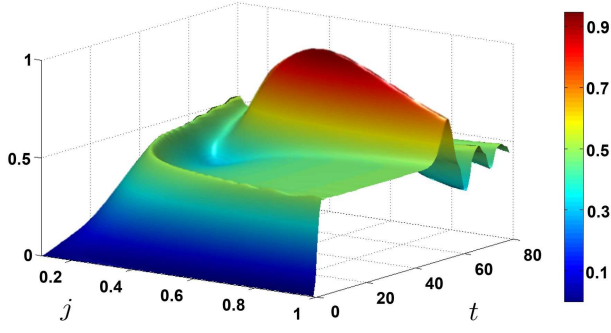


FIG. 4: Time evolution of the minimum fidelity $\mathcal{F}_{AB}^{\min}(t)$ vs j , for a $N=50$, $\gamma=\gamma_0=\gamma_N=0$ (XX chain), and $h=h_Q=0$.

the quality of the entanglement transmission from AA' to BA' . Indeed, this can be done using the following lower bound [17, 18] for the fidelity of entanglement: $\mathcal{F}(|\psi_{AA'}\rangle\langle\psi_{AA'}|, \rho_{BA'}(t)) \geq \frac{3}{2}\mathcal{F}_{AB}^{\min}(t) - \frac{1}{2}$. The entangled state $|\Psi_{AA'}\rangle$ can be hereafter taken as any of the Bell states, since a local operation on A' does not change the dynamics of the concurrence.

The above quantities essentially depend on the time evolution of the magnetizations of B, which we have determined as follows: The components S_{N+1}^α have been written in terms of the fermionic operators $\{\eta_k, \eta_k^\dagger\}$; their time-dependence, which simply follows from $\mathcal{H} = \sum_k \omega_k \eta_k \eta_k^\dagger$, has been made explicit, so as to obtain, by transforming back via inverse Bogolubov and Jordan-Wigner transformation, the Heisenberg representation of the spin operators $S_{N+1}^\alpha(t)$. Their expectation values, i.e., the required magnetizations, have been finally derived by expanding $S_{N+1}^\alpha(t)$ in terms of the spin operators of the extremal qubits and of fermionic operators relative to the wire, i.e., defined by the diagonalization of the sole \mathcal{H}_r . This approach allows us to devise a numerical procedure for determining, for any model belonging to the class described by the Hamiltonian (3), the Kraus operators [17] $M_\mu(t)$, in terms of which we get $\rho_B(t) = \sum_\mu M_\mu(t) \rho_A(0) M_\mu^\dagger(t)$ and $\rho_{BA'}(t) = \sum_\mu [M_\mu(t) \otimes 1] \rho_{AA'}(0) [M_\mu^\dagger(t) \otimes 1]$, i.e., the necessary tools for our analysis.

We first consider the fidelity $\mathcal{F}_{AB}(t)$ as defined above. In the case of the XX chain, we put ourselves in the worst possible case and evaluate $\mathcal{F}_{AB}^{\min}(t)$: a high value of such quantity ensures a very good transfer of *any* initial state, modulo a local operation. In Fig. 4 we show $\mathcal{F}_{AB}^{\min}(t)$ for the same model as in Fig. 3 as a function of the coupling j . A minimum fidelity which is just slightly below unity is observed for that same value of the coupling, determined by our procedure, which gives rise to the dynamics shown in the lower panel of Fig. 3, i.e., $j = j_{\text{opt}} \simeq 0.58$.

We underline that such a high value of the minimum fidelity, which is confirmed also for N as large as 500, does also imply an optimal transmission of entanglement. It is worth noticing that the peak of $\mathcal{F}_{AB}^{\min}(t)$ occurs simultaneously with that of $\langle S_B^z(t) \rangle$, i.e., at $t \simeq N$, a feature

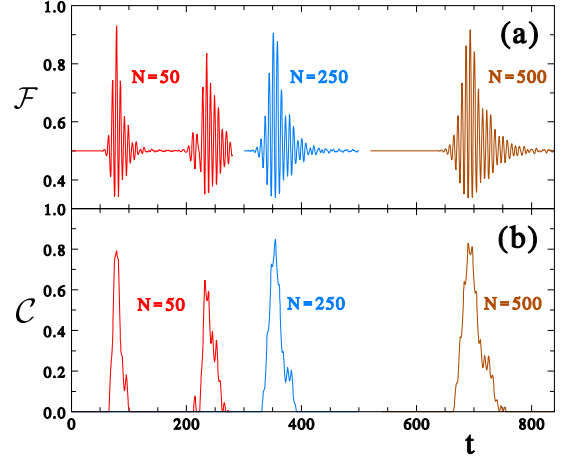


FIG. 5: Time evolution of $\bar{\mathcal{F}}_{AB}$ (upper panel) and $\mathcal{C}_{BA'}(t)$ (lower panel), for different N , $\gamma=\gamma_0=0.5$, $h=0.5$, $h_Q=0.85$, and $j_{\text{opt}}=0.49, 0.39, 0.34$ for $N=50, 250, 500$, respectively. The second signal observed in both panels is the first echo for $N=50$.

that can be analytically proven in the XX case.

The consistence of the above picture allows us to move forward to the analysis of the entanglement transfer, and also in the more general case of the XY model. We set $\gamma=\gamma_0=0.5$ and $h=0.5$ and apply our procedure: We first locate the inflection point of the dispersion relation in the thermodynamic limit at $k_0 \simeq 1.795$ and tune $h_Q \simeq 0.85$. This latter value slightly differs from ω_{k_0} as determined in the thermodynamic limit, due to finite-size and boundary effects. Finally, Eq. (2) allows us to numerically determine $j_{\text{opt}} = 0.49, 0.39, 0.34$ for $N=50, 250, 500$, respectively.

Despite the dynamics in the XY case being complicated by the existence of several energy scales, we manage to get a very good state transfer also for large N , as testified by the fidelity, averaged over all possible initial states of A, $\bar{\mathcal{F}}_{AB}$ shown in the upper panel of Fig. 5. The oscillations observed are due to the precession induced by the magnetic field h_Q acting on the qubit B. Such precession causes only a phase change and does not affect the entanglement transfer, as shown below. The entanglement between B and A' is monitored by the time evolution of the concurrence $\mathcal{C}_{BA'}(t)$, shown in the lower panel of Fig. 5 for different values of N : For each N , this is characterized by an extremely well defined peak, of height > 0.8 even for chains as long as 500 sites. This peak occurs simultaneously with the maximum of $\bar{\mathcal{F}}_{AB}$ and has a finite but small width.

The second signal observed in both panels is the first echo for $N=50$, which is seen to be still considerably intense and well localized: this testifies of a long-lasting quasi non dispersive dynamics of the wire. Other echoes occur at times which are too large for being shown in the figure, but their structure confirm this statement.

In conclusion, based on a general picture of the dynamical evolution of quantum wavepackets, we have devised

a procedure for inducing an optimal dynamics in a quantum wire. The procedure leads to the determination of an optimal coupling between the wire and the two qubits attached at its endpoints, meanwhile giving indications about the best setting of other, possibly tunable, parameters of the system. For the procedure to apply, few very simple conditions must be fulfilled, and there is no need for a specific design neither of the wire, nor of its initial state. By implementing our approach to the spin- $\frac{1}{2}$ XY chain, we have obtained extremely good quantum-state and entanglement transfer. The time scales over which such transfer is found to occur is considerably shorter

as compared with previous results concerning quantum-state transmission over Heisenberg models [1, 15, 19–21]. Moreover, the quality of the state and entanglement transfer that we obtain very weakly deteriorate as the length of the wire increases.

PV gratefully thanks Dr. N. Gidopoulos for useful discussions, and the ISIS centre of the Science and Technology Facilities Council (UK) for the kind hospitality. PV also acknowledges financial support from the Italian CNR under the "Short-term mobility 2010" funding scheme.

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- [1] S. Bose, *Contemp. Phys.* **48**, 13 (2007).
 - [2] S. Bose, *Phys. Rev. Lett.* **91**, 207901 (2003).
 - [3] T. J. Osborne and N. Linden, *Phys. Rev. A* **69**, 052315 (2004).
 - [4] M. Christandl, N. Datta, T. C. Dorlas, A. Eckert, A. Kay, A. Langdahl, *Phys. Rev. A* **71**, 032312 (2005).
 - [5] H. L. Haselgrove, *Phys. Rev. A* **72**, 062326 (2005).
 - [6] L. Campos Venuti, C. Degli Esposti Boschi, and M. Roncaglia *Phys. Rev. Lett.* **99**, 060401 (2007).
 - [7] C. Di Franco, M. Paternostro, M. S. Kim, *Phys. Rev. Lett.* **101**, 230502 (2008).
 - [8] S. Yang, Z. Song, C. P. Sun, *Sci. China Ser. G* **51**, 45 (2008).
 - [9] T. S. Cubitt and J. I. Cirac, *Phys. Rev. Lett.* **100**, 180406 (2008).
 - [10] M. Miyagi and S. Nishida, *Appl. Optics* **18**, 678 and 2237 (1979).
 - [11] If the dispersion relation ω_k does not display a wide interval of linearity, the wire is unlikely to behave as a good quantum channel, at least by the mechanism considered here.
 - [12] S. Paganelli, G. L. Giorgi, and F. de Pasquale, *Fortsch. Physik* **57**, 1094 (2009).
 - [13] C. A. Bishop, Y-C. Ou, Z-M. Wang, and M. S. Byrd, *Phys. Rev. A* **81**, 042313 (2010).
 - [14] E. H. Lieb, T. Schulz and D. Mattis, *Ann. Phys.* **16**, 407 (1961).
 - [15] A. WóJcik, T. Łuczak, P. Kurzyński, A. Grudka, T. Gdala, M. Bednarska, *Phys. Rev. A* **72**, 034303 (2005).
 - [16] $\mathcal{C}(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$ where λ_i are the eigenvalues of $\sqrt{\sqrt{\rho}(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)\sqrt{\rho}}$.
 - [17] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
 - [18] E. Knill, R. Laflamme, *Phys. Rev. A* **55**, 900 (1997).
 - [19] F. Plastina and T. J. G. Apollaro, *Phys. Rev. Lett.* **99**, 177210 (2007).
 - [20] G. Gualdi, V. Kostak, I. Marzoli, and P. Tombesi, *Phys. Rev. A* **78**, 022325 (2008).
 - [21] S. I. Doronin and A. I. Zenchuk, *Phys. Rev. A* **81**, 022321 (2010).